$$\frac{2}{\gamma+1} \cdot \frac{1}{\gamma^{1/2}} \cdot \frac{M^2}{\bar{\chi}_t} \left(\frac{\epsilon k}{\alpha^5} \right)^{1/2} C_H = \frac{0.332(1+0.382/\xi)}{\xi^{1/4}(1.145+\xi)^{1/2}} \tag{10b}$$

where

$$\xi \equiv \alpha^2 \left(\frac{\gamma+1}{2} \cdot \frac{x}{\epsilon k t} \right)^{2/3} \tag{11}$$

The notation is as defined by Kemp.[2] The unbroken curves in Figs. 2 and 3 represent Eqs. (10a) and (10b), respectively. The agreement with the experimental data is similar to that shown in Kemp's[2] original Fig. 10. In the region $\xi \simeq 1$, where the pressure curve in Fig. 2 is leveling out, the agreement in both pressure and heat transfer is better than is obtained with the exact solution of the Cheng equation (shown by the dashed curves in Figs. 2 and 3).

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Boundary-Layer Effect in Panel Flutter

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Nomenclature

- $a(y)$ = speed of sound nondimensionalized by a_∞
- a_∞ = speed of sound in mainstream
- $A_\pm$ = defined by Eq. (20)
- b = $k^2/2ik_1 M_\infty U'$
- $B_\pm^{(r)}$ = defined by Eq. (21)
- $C_\pm, C_0$ = arbitrary constants
- d = plate semi-length
- $\hat{i}$ = unit vector in x-direction
- $\hat{k}$ = unit vector in z-direction
- $\boldsymbol{k}$ = $\hat{i}k_1 + \hat{k}k_3$ wave number vector
- k_1 = wave number in x-direction
- k_3 = wave number in z-direction
- k = $|\boldsymbol{k}|$
- K = defined by Eq. (23)
- M_∞ = freestream Mach number $U_\infty/a_\infty > 1$
- $\mathfrak{M}$ = $M_\infty[(\omega/k_1) - U]/a$
- p = pressure fluctuation
- P = Fourier transfer of p
- t = time, nondimensionalized by U_∞/d
- $U(y)$ = mean boundary layer velocity at y
- U_p = $U(\bar{y}_p)$
- U_∞ = mean velocity in mainstream
- w_p = plate deflection
- W_p = Fourier transform of w_p
- x,y,z = spatial coordinates nondimensionalized by $d/2$
- $\bar{y}_p$ = effective mean wall position
- α = $[2iM_\infty/(U'k_1)]^{1/2}$
- ∇ = gradient operator
- β = $[M_\infty^2(\omega-k_1)^2 - k^2]^{1/2}$
- γ = ratio of specific heats
- Γ = defined in Eq. (12)
- δ = $\Delta/(d/2)$
- Δ = boundary-layer thickness
- ξ = $[2ik_1/(U'M_\infty]^{1/2}$
- $\bar{\rho}_p$ = average density at plate
- ω = dimensionless frequency

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Introduction

As pointed out by Dowell,[1] it is now firmly established that the adjacent boundary layer will have an important influence of the flutter behavior of plates. This effect has been considered by a number of investigators.[1-3] An important part of the problem (the aerodynamic part) is the determination of the relation between the fluctuating force exerted by the flow on the panel and the transverse displacement of the panel. (This relation is frequently expressed as a "generalized aerodynamic force.") It usually is necessary to determine the resulting force numerically.[1-3] In this Note we shall show that if the supersonic Mach number of the stream is not too large, an analytical expression can be obtained for this force. The low supersonic Mach numbers are the ones of maximum interest in the present problem because it is in this Mach number region where the boundary layer has the most influence. For example, Dowell[1] shows that the presence of the boundary layer causes about a 300% increase in flutter dynamic pressure at a Mach number of about 1.2, while it causes only about a 20% increase at a Mach number of 2.

Analysis

The configuration of interest is shown in Fig. 1. It can be shown by rearranging Eq. (18) of Ref. 1 that the pressure fluctuations in the boundary layer caused by the motion of the panel is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{a^2}\nabla\cdot(a^2\nabla p) - \frac{M_\infty^2}{a^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}p\right] - 2U'\frac{\partial^2 p}{\partial x \partial y} = 0 \tag{1}$$

where all lengths are nondimensionalized by the panel half length d, the time t is nondimensionalized by U_∞/d, the mean velocity is nondimensionalized by U_∞, the speed of sound is nondimensionalized by a_∞,

$$(D/Dt) \equiv (\partial/\partial t) + U(\partial/\partial x) \tag{2}$$

and the prime denotes differentiation with respect to y.

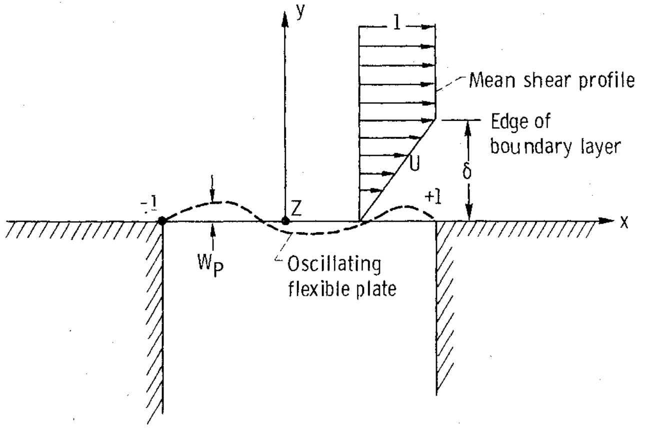

Fig. 1 Geometry of problem.

Following Dowell[1] we suppose that the effect of the recovery factor can be neglected so that we can write

$$a^2 \approx 1 + ((\gamma - 1)/2) M_\infty^2 \; (1 - U^2) \tag{3}$$

We suppose that the mean velocity varies only in the y direction. The boundary condition at the wall is

$$\frac{\partial p}{\partial y} = -\bar{\rho}_p U_\infty^2 \left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right]^2 w_p \qquad y = \bar{y}_p \tag{4}$$

where (for reasons explained in detail by Dowell[1]) $\bar{y}_p$ is taken to be some effective position of the plate rather than zero [hence U is not zero in Eq. (4)] and w_p is the plate displacement, which will be taken as zero for $|x| < 1$.

For flutter we are only interested in a harmonic $e^{-i\omega t}$ time dependence. The problem can then be simplified by expressing p and w_p in terms of the two dimensional Fourier transforms $P(y,\boldsymbol{k})$ and $W_p(\boldsymbol{k})$, respectively, by

$$p = e^{-i\omega t} \int\!\!\int e^{i(k_1 x + k_3 z)} P(y,\boldsymbol{k}) \, \mathrm{d}\boldsymbol{k} \tag{5a}$$

$$w_p = e^{-i\omega t} \int\!\!\int e^{i(k_1 x + k_3 z)} W_p(\boldsymbol{k}) \, \mathrm{d}\boldsymbol{k} \tag{5b}$$

where $\boldsymbol{k} = \hat{i} k_1 + \hat{k} k_3$ is the wave number vector. Then Eq. (1) reduces to the equation

$$\mathfrak{M}^2 (P'/\mathfrak{M}^2)' + (k_1^2 \mathfrak{M}^2 - k^2) P = 0 \tag{6}$$

where $k = |\boldsymbol{k}|$,

$$\mathfrak{M} \equiv M_\infty [(\omega / k_1) - U] / a \tag{7}$$

and the boundary condition (4) becomes

$$(\mathrm{d}P/\mathrm{d}y) = U_\infty^2 \bar{\rho}_p k_1^2 [(\omega/k_1) - U]^2 W_p \qquad y = \bar{y}_p \tag{8}$$

The case of principal interest is where $\gamma \approx 1.4$ (air). Then for $M_\infty \leq 1.2$, $|a^2 - 1| \leq 0.3$ Hence at low supersonic Mach numbers we should not introduce much error by assuming that $a \approx 1$. Moreover we suppose that the effect of the boundary layer can be approximated by assuming a linear profile‡

$$U = U' y \tag{9}$$

where

$$U' = 1/\delta = d/\Delta = \text{constant} \tag{10}$$

Calculations[1-3] indicate that the fluctuating surface forces are relatively insensitive to the exact shape of the mean velocity profile. In fact, for a similar problem involving acoustic propagation, Mariano[4] has pointed out the linear velocity profile results are almost indistinguishable from those obtained with a 1/7th law turbulent velocity profile.

With these approximations we can now solve Eq. (6) by using the method of Ref. 5. To this end we introduce the new independent variable ξ and the new dependent variable Γ by

$$\xi = (2ik_1 / U' M_\infty)^{1/2} \mathfrak{M} \tag{11}$$

and

$$P = [e^{b\xi^2/2} / (b + 1/2)] \frac{\mathrm{d}}{\mathrm{d}\xi} (e^{-b\xi^2/2} \Gamma) \tag{12}$$

where we have put

$$b = \frac{k^2}{2ik_1 M_\infty U'} \tag{13}$$

‡In this case, Eq.(6) will have a regular singular point at $\mathfrak{M} = 0$. However, the solutions which we obtain exist and are valid at this point.

Then Eq. (6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left\{ \frac{e^{-b\xi^2/2}}{\xi^2} [\Gamma'' - (1/4 \xi^2 + b)\Gamma] \right\} = 0$$

where the primes now denote differentiation with respect to ξ. The first integration in this equation can easily be carried out and without loss of generality the resulting constant of integration may be set equal to zero. But this shows that Γ satisfies Weber's equation[5,6,7]

$$\Gamma'' - (1/4 \xi^2 + b)\Gamma = 0$$

whose general solution is an arbitrary linear combination of Weber[7] functions, $U(b, \pm\xi) \equiv D_{-b-(1/2)} (\pm\xi)$. Hence the solution to Eq. (6) is an arbitrary linear combination

$$P = C_+ P^+ + C_- P^- \tag{14}$$

of the functions

$$\begin{aligned} P^\pm(\xi) &= \pm \frac{e^{b\xi^2/2}}{(b + 1/2)} \frac{\mathrm{d}}{\mathrm{d}\xi} e^{-b\xi^2/2} U(b, \pm\xi) \\ &= \frac{U'(b, \pm\xi) \mp b\xi U(b, \pm\xi)}{b + 1/2} \\ &= -U(b-1, \pm\xi) + (b - 1/2) U(b+1, \pm\xi) \end{aligned} \tag{15}$$

where the last form was obtained by using the recurrence relations for the parabolic cylinder functions.[7] These recurrence relations can also be used to show that

$$\frac{\mathrm{d}P^\pm(\xi)}{\mathrm{d}\xi} = 1/2 \xi [U(b-1, \pm\xi) + (b - 1/2) U(b+1, \pm\xi)] \tag{16}$$

The constants in Eq. (14) must be determined in such a way that the solution satisfies the boundary condition (8) at the plate and behaves like an outgoing wave at infinity. We shall consider the latter condition first. Thus for $y > \delta$, U is a constant, and Eq. (6) becomes

$$P'' + \beta^2 P = 0 \tag{17}$$

where

$$\beta \equiv [M_\infty^2 (\omega - k_1)^2 - k^2]^{1/2}$$

The solution to this equation which is either bounded at infinity or behaves like an outgoing wave is

$$P = C_0 e^{-i\beta y}$$

where C_0 is an arbitrary constant and the branch cuts for β are chosen so that they lie in the upper half complex k_1-plane. Then since P and P' must be continuous at the outer edge of the boundary layer, the solution (14) will match the solution (17) provided the former satisfies the "radiation" condition

$$(\mathrm{d}P/\mathrm{d}y) + i\beta P = 0 \qquad \text{at } y = \delta \tag{18}$$

Inserting the solution (14) into the boundary conditions (8) and (18), eliminating $C_\pm$, and using Eq. (16) now shows that

$$P = \left[\frac{\bar{\rho}_p U_\infty^2 W_p (\omega - U_p k_1)}{M_\infty} \right] \frac{A_- P^+(\xi) - A_+ P^-(\xi)}{A_- B_+^{(0)} - A_+ B_-^{(0)}} \tag{19}$$

where we have put

$$\begin{aligned} A_\pm \equiv {} & [(\omega - k_1) M_\infty + \beta] U(b-1, \pm\alpha(\omega - k_1)) \\ & + (b - 1/2) [(\omega - k_1) M_\infty - \beta] U(b+1, \pm\alpha(\omega - k_1)) \end{aligned} \tag{20}$$

$$B_{\pm}^{(r)} \equiv (-1)^r U(b-1, \pm\alpha(\omega - U_p k_1)) + (b-\tfrac{1}{2}) U(b+1, \pm\alpha(\omega - U_p k_1)) \text{ for } r=0,1 \tag{21a}$$

$$U_p \equiv U(\bar{y}_p) \tag{21b}$$

$$\alpha \equiv (2iM_\infty / U' k_1)^{1/2} \tag{21c}$$

This result shows that the pressure fluctuation at the plate surface $P(\bar{y}_p, k)$ is related to the plate displacement W_p by

$$\frac{P(\bar{y}_p, k)}{\bar{\rho}_p U_\infty^2} = \frac{\omega - U_p k_1}{M_\infty} \frac{A_- B_+^{(1)} - A_+ B_-^{(1)}}{A_- B_+^{(0)} - A_+ B_-^{(0)}} W_p \tag{22}$$

This is the desired result. When it is inserted into Eq. (5a) we obtain through the use of Eq. (5b)

$$\frac{p(x,o,z,t)}{\bar{\rho}_p U_\infty a_\infty} = \left[\frac{\partial}{\partial t} + U_p \frac{\partial}{\partial x}\right] \times \iint_{\text{plate surf.}} K(x-x', z-z') w_p(x', z', t) \, dx' dz'$$

where

$$K(x,z) \equiv \frac{i}{(2\pi)^2} \iint e^{i(k_1 x + k_3 z)} \times \left[\frac{A_- B_+^{(1)} - A_+ B_-^{(1)}}{A_- B_+^{(0)} - A_+ B_-^{(0)}}\right] dk$$

In applications one often deals with "generalized aerodynamic forces" rather than actual plate pressure. Expressions for these forces can be obtained most directly by inserting Eq. (22) into the formulas given by Dowell.[1]

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Stability of Beam-Reinforced Circular Plates

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CIRCULAR plates loaded in compression are often employed as structural components in engineering systems. Several authors have investigated the elastic stability of circular plates subjected to various loadings and boundary conditions. G. H. Bryan[1] discussed the stability of a planar plate under in-plane compression and presented the solution to the particular problem of a clamped, solid circular plate under a normal, uniform compressive load per unit length of the plate circumference. He employed energy principles and applied a variational method to obtain his solution. Bryan considered symmetric and nonsymmetric buckling modes in his analysis. H. Reismann[2] solved the problem of the buckling of a solid circular plate with an elastic, spring-type, boundary restraint along the plate circumference. Reismann assumed that the buckling load was associated with the radially symmetric mode. Amon and Widera[3] further developed the Reismann case and analyzed the stability of a solid circular plate with a surrounding edge beam of rectangular cross section. These authors also assumed that the plate buckled into the radially symmetric mode.

In the work done by Yamaki,[4] considering the buckling of annular plates under uniform compression on both plate edges, it was found that higher modal shapes often are associated with the critical buckling loads. Phillips and Carney[5] have recently obtained closed form solutions to show that higher buckling modes can be critical in the case of annular plates with edge beams.

In this note, the edge beam and plate configuration shown in Fig. 1 is allowed to buckle into any mode, so that all possible buckled shapes are considered in order to obtain the lowest, or "critical" buckling load.

Theory

The edge beam in Fig. 1 is simply supported along its entire length, as indicated. The support used under the edge beam constrains the beam to remain in its original plane. It is assumed in the analysis that the plate is integrally attached to the edge beam at the outer edge of the plate. The external load that is applied to the structure P_o is a uniformly distributed compressive load per unit of beam length applied to the outer side of the edge beam.

The solid circular plate has a radius a and a constant thickness t. The other material properties which are pertinent to this analysis are A_B the edge beam cross-sectional area E_B the modulus of elasticity of the beam material, I_B the moment of inertia about the horizontal centroidal axis of the beam cross section, I_{20} the moment of inertia about the vertical centroidal axis of the beam cross section, E the modulus of elasticity of the plate material, and σ Poisson's ratio for the plate material.

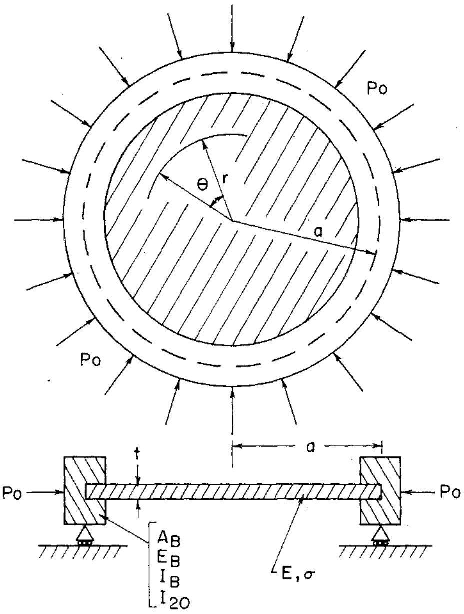

Fig. 1 Plate-edge beam configuration.

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